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Buffon's cycloid

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SUMMARY

In Buffon's account of his famous problem of the needle, it is not clear how he actually obtained the answer. A conjecture is put forward in this note, based on Buffon's apparent mathematical abilities and inclinations, insofar as they can be determined from his correspondence.

Some key words: Area of cycloid; Buffon's needle; History of probability; Geometrical probability.

A recent note by Stonebridge (1979), an expository article by Gani (1980) and a survey of ecological applications of line intersect sampling (De Vries, 1979, p. 9) have all drawn attention to the geometrical presentation of the solution of Buffon's needle problem. It is therefore interesting to note that Buffon himself used a geometrical method in deriving his solution. It was not however that employed by the writers noticed above, and it has perplexed some commentators on Buffon's work (Coolidge, 1949, p. 175; Hanks, 1966, p. 51). This paper offers a conjecture as to how Buffon obtained his solution.

George Louis Leclerc, later Comte de Buffon, presented to the Royal Academy of Science, Paris, a paper in which he described and solved a number of problems in geometrical probability, thus founding this branch of mathematics. It was read by Clairaut at the session on May 6th 1733. We have the referees' report (Maupertius and Clairaut, 1733) and the résumé of the session (Fontenelle, 1733), but the paper was not published in full. However, these contemporary reports make it clear that the text was essentially the same as the section on geometrical probability (pp. 95–105) in the *Essai d'Arithmétique Morale* (Buffon, 1777) published more than forty years later; see Hanks (1966, p. 12).

Buffon began by studying the 'Jeu du franc-carreau' where a circular coin radius b is thrown onto a floor on which is marked a square grid of side $2a$. He noted that the cases favourable to the event that the coin falls 'franc', that is in a single square, are those where its centre falls within a smaller inscribed square of side $2a–2b$, while those favourable to its hitting two or more squares correspond similarly to the 'corona' surrounding the inscribed square. After giving the ratio $a:b = 1:1-(\frac{1}{2})^{\frac{1}{2}}$ needed to make the probabilities equal, Buffon considered several generalizations. He listed answers, not all correct, to the corresponding problem where the square grid is replaced by a grid of triangles, lozenges or hexagons, calculating not only the probability of falling 'franc' but of hitting specified numbers of tiles. He mentioned the possibility of replacing the circular coin by a square one, or by a needle.

Next, Buffon analysed the problem for which his name is known to probability theorists. As a temporary simplification, he replaced the square grid by a single set of parallel lines with a spacing of $2a$. The length of the needle, $2b$, is taken to be less than $2a$.

Buffon noted that by symmetry, we can confine ourselves to cases where ε, the centre of the needle, falls in the top of the strip between two neighbouring parallels, and when one of its ends falls in the north east quadrant.

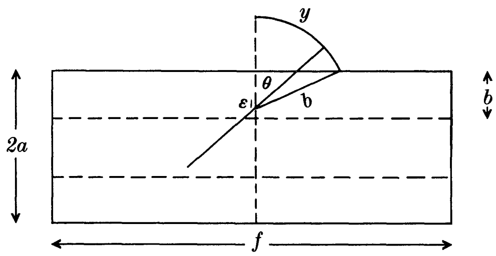

Fig. 1. The unbroken horizontals are the parallels. The needle, centre ε, is in a 'hit' position.

In modern notation and terminology, we may either condition on the distance x of ε from the nearest parallel, and note that a hit occurs if this is less than b, and the angle θ of the needle to the verticle satisfies $\theta \leqslant \cos^{-1}(x/b)$. Alternatively we may condition on θ and note that a hit is obtained if $x \leqslant b\cos\theta$. Integrating over the distributions of the conditioning variables, we obtain respectively

$$\frac{1}{a}\int_0^b \frac{2}{\pi}\cos^{-1}\left(\frac{x}{b}\right)dx = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi}\frac{b\cos\theta}{a}d\theta = \frac{2b}{\pi a}.$$

Buffon followed the first approach, but as an 18th century man he worked in terms of arcs rather than angles. Thus, given $x \leqslant b$ he measures the number of cases favourable to a $\int y\,dx$, while all possible cases are measured by fac, where f is the length of a section of the strip between two parallels and $c = \frac{1}{2}\pi b$ is the arc of a quarter circle of radius b. He continues:

'If then we want the game to be equal, we will have $ac = \int y\,dx/(\frac{1}{2}c)$, that is to say to the area of part of the cycloid whose generating circle has diameter $2b$, the length of the baton. Now we know that this area of the cycloid is equal to the square of the radius, thus $a = b^2(\frac{1}{2}c)$, . . .''

Buffon's lack of verbal precision has been noted by Hanks (1966, pp. 16, 51) and it is $\int y\,dx$ which is the area of the part of the cycloid. He does not say which area is involved, but it is the shaded part in Fig. 2.

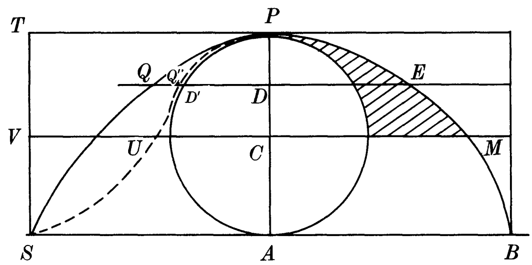

Fig. 2. The cycloid and its generating circle. The shaded area equals Buffon's integral.

Coolidge (1949, p. 175) says dismissively of this reasoning

'It does not seem that Buffon was quite up to integrating $\cos^{-1}(x/b)$'.

Hanks (1966, p. 51) comments:

'That is perhaps unjust. This little astuteness would seem very elegant to Buffon's contemporaries. The memoir does not express y in terms of x: we ask in vain how Buffon saw that the integral is the same as for the cycloid. Is it that the cycloid was so familiar that it could serve as an explanation in this way?'

We are faced then with three problems. (i) Could Buffon evaluate integrals of the type involved by analytic means? (ii) How did he see that the integral was the area under a cycloid? (iii) How did he know what that area was?

(i) Buffon's mathematical work has been discussed by Brunet (1936), Fréchet (1952), Hanks (1966, Chs. 1–3) and Roger (1977). Buffon was more interested in the philosophical background to mathematics, particularly in relation to its applications, than in either the abstract theory or techniques for solving problems. When he discussed the latter, he usually relied on verbal reasoning rather than analytic calculations. This is true both of the *Essai d'Arithmétique Morale* and of the *Préface* to his translation of Isaac Newton's *Method of Fluxions* (Buffon, 1740). Apart from his work on the needle problem, there is very little evidence about Buffon's specific mathematical ability, but what there is indicates that it was not very high. The main source is in his correspondence with Gabriel Cramer (1701–52), Professor of Philosophy and Mathematics at Geneva from 1721.

Twelve letters from Buffon to Cramer were found in the library of the University of Geneva and published by Weil (1961), together with five already known. The first four letters were written in 1731 after which there was a five year break in the correspondence. In them Buffon discusses mathematical problems put to him by Cramer. An examination of this correspondence gives us an idea of the relative strengths and weaknesses of Buffon's mathematics. In letter 2 (24th July 1731) he refers (Weil, 1961, pp. 112–3) to a problem requiring the equation of a certain locus, and admits that he finds it too difficult to make any progress. In letter 4 (25th November 1731) he enunciates the problem and gives the solution found by Perelli, of Florence, whom he had recently visited (Weil, 1961, pp. 118–20). The tone of this letter suggests that although he followed Perelli's solution, which involved using the geometrical conditions to write down two differential equations and to eliminate a redundant variable, then making a geometrically meaningful change of variables so that the new ones separated out, he could not have derived it himself. On the other hand, he seems very confident in a problem of statics discussed in letter 2 (Weil, 1961, pp. 113–4). He employs a fully geometrical argument to find the resultant of a continuously distributed force, and he exploits the comparison between his solution and Cramer's to derive a way of finding centres of gravity.

Buffon appears to be successful when using methods involving geometrical arguments that can be seen in concrete form, but unable to use the analytic methods of the calculus for himself.

(ii) The proof that that area under the cycloid, $PABP$ in Fig. 2 is $\frac{3}{2}\pi b^2$ was found by Roberval, probably soon after 1637. His work is studied by Auger (1962, pp. 39–49). The method hinges on the companion curve $PQ'S$ defined by $DD' = QQ'$ for a general horizontal which is readily seen to bisect the rectangle $PAST$ whose area is $2\pi b^2$. On summing the infinitesimal horizontal strips swept out by DD' and QQ', we see that the areas in half the generating circle and in the 'space between the cycloid and its companion', cut off by a general horizontal, are equal. Use of the whole of these areas gives Roberval's result.

Next, referring to Fig. 3, suppose that while the generating circle rolls to the right, another circle with centre fixed at C rotates with the same angular velocity. The point A, to which C is displaced on this circle will lie vertically above C', the centre of the rolling circle. Hence $C'N = CC' = x = RR'$, the linear displacement of the rolling circle. Thus $\operatorname{arc} QR'$ in Fig. 3 is equal to $\operatorname{arc} y$ in Fig. 1, when ε is at a distance x from the nearest parallel. Thus the area in Fig. 3 corresponding to the shaded area in Fig. 2 is made up of infinitesimal annular arcs of area $(\operatorname{arc} QR')\,\delta x$, which correspond to the number of cases

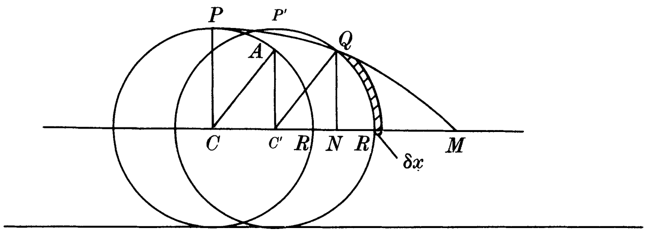

Fig. 3. The semicycloid above its line of centres of the generating circle. The shaded strip gives an identifiable subset of 'favourable cases'.

favourable to the needle crossing a parallel, given x. Hence the total shaded area measures the total number of favourable cases. Note that in the figure of Hanks (1966, p. 55) the generating circle is shown disjoint from its initial position, which prevents the construction given with reference to Fig. 3. This cannot however occur while the relevant segment of the arc is being generated.

My conjecture is that Buffon used some such reasoning as this to obtain the number of favourable cases. His work came at the end of 150 years of development in which the evaluation of areas and volumes by mapping them, infinitesimal strip by strip onto known areas or volumes, had given way to the analytic methods of the calculus. However, Buffon was an amateur, and the reasonable assumption that he may have been more at home with the 'older' methods than the 'new' ones is supported by the extracts from his correspondence with Cramer. The idea of displaying the total number of favourable cases as a concrete, measurable area is in keeping with his procedure in dealing with the simpler cases of the 'jeu du franc carreau' and accords with his well attested preference for the real, in philosophical matters generally and in particular in mathematics; see the works cited at the beginning of (i). Furthermore, the mapping of infinitesimal strips from one part of a figure to another is precisely the method used by Roberval in his basic quadrature of the cycloid.

It is not even necessary to suppose that Buffon was capable of the detailed geometrical argument used to establish the equivalence discussed above, since the fact that as the generating circle rolls, the portion of its arc lying in the shaded area decreases from $\frac{1}{2}\pi b$ to 0, in a manner similar to the way that the arc corresponding to a hit decreases in Fig. 1 as x increases from 0 to b, leads to a strong intuitive feeling that the area under the cycloid represents the integral.

The calculation of an area as the sum of infinitesimal strips other than vertical trapezoids would be much more familiar to Buffon than it is to us. Pascal (1659) had used such a method in studying the cycloid, while Johann Bernoulli, whose works were published by Cramer in 1742, discussed them in his work on integral calculus.

(iii) The shaded area in Fig. 2 required by Buffon can be deduced from results given by Roberval (1693), combining Propositio Sexto, p. 255, with a result given in the section *Ad primum, sequens notandam* on p. 260. Moreover, a remark by Pascal (1658, p. 181) implies that the result was widely known at that time. I hope in the future to discuss the possible sources of Buffon's technical knowledge of mathematics in more detail, elsewhere.

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